Lecture 29 (12/3/21)

- Ex ① from Lecture 28 notes.

Ex. ② Let f be analytic in G s.t. $\overline{B(0,1)} \to \overline{\mathbb{D}} \subseteq G$. Assume $|f|<1$ on $|z|=1$. Show that f has a unique fixed point, $f(z)=z$, in $\mathbb{D}$

We apply Rouché's Thm to count zeros of $g(z)=f(z)-z$. For this, we compare g to $h(z)=z$, which of course has precisely one zero in $\mathbb{D}$: For $|z|=1$,

$$|g(z)+h(z)| = |f(z)| < 1 = |z| \leq |g(z)|+|h(z)|.$$

The conclusion now follows from Rouché.

If $|f| \leq 1$ on $|z|=1$, then $f(z)$ could be $f(z)=z$ (every pt in a fixed point) or $f(z)=1$ (the only fixed point is $z=1$, not in $\mathbb{D}$.

③ Find all (if any) entire f s.t. $|f|=1$ on $|z|=1$ and $f''(0)<0$.

For $a \in \mathbb{D}$, consider the Möbius transf.

$\varphi_a = \frac{z-a}{1-\bar{a}z}$. It satisfies

$$|\varphi(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right| = \frac{|e^{i\theta}-a|}{|e^{-i\theta}-\bar{a}|}$$

$$= \frac{|e^{i\theta}-a|}{|\overline{e^{i\theta}-a}|} = 1 \text{ and hence}$$

$|\varphi(z)|=1$ on $|z|=1$. Moreover, φ_a has simple zero at $z=a$ and simple pole at $z=1/\bar{a}$. Now, let $a_1, a_2, \dots, a_m$ be zeros of f in $\mathbb{D}$ w/ multi (note there can only be finitely many since $|f|=1$ on $|z|=1$). Then

$$g(z) = \frac{f}{\varphi_{a_1} \cdots \varphi_{a_m}}$$ has no zeros

in $\mathbb{D}$, $|g|=1$ on $|z|=1$, and g is entire, since it only has removable sing. (by constr.) at the points $a_1, \ldots, a_m$, and has no zeros in $\overline{\mathbb{D}}$. By Max Mod Princ, $|g| \leq 1$ in $\overline{\mathbb{D}}$. Since $1/g$ is also analytic in $\mathbb{D}$, $|1/g| = 1$ on $|z|=1$, MMP $\Rightarrow$ $|g| \geq 1$ in $\overline{\mathbb{D}}$ $\Rightarrow$ $|g|=1$ in $\overline{\mathbb{D}}$. By Open Mapping Thm, $g(z)$ is constant $= c$ s.t. $|c|=1$.

$\Rightarrow$ $f(z) = c\, \varphi_{a_1}(z) \ldots \varphi_{a_m}(z)$.

But since each φ_{a_j} has pole at $z = 1/\overline{a_j} \in \mathbb{C} \setminus \overline{\mathbb{D}}$ <u>unless</u> $a_j = 0$, we conclude that f can only be entire when each $a_j = 0$ $\Rightarrow$

$$f(z) = c\, z^m.$$

$$\Rightarrow \quad f' = mcz^{m-1}, \quad f'' = m(m-1)cz^{m-2}$$

$$\Rightarrow \quad f''(0) = \begin{cases} 0, & m \neq 2 \\ 2c, & m = 2. \end{cases}$$

Thus, $f''(0) < 0 \Rightarrow m = 2, \; c = -1,$

i.e. $f(z) = -z^2$.